On the Computation of Determinants Arising in Some Bivariate Rational Interpolation Problems

M. Gasca and J. J. Martínez*
Departamento de Matemática Aplicada
Universidad de Zaragoza
Zaragoza, Spain

ABSTRACT

Bivariate Cauchy-Vandermonde determinants arise in some bivariate rational interpolation problems. These determinants are easily computed by means of the technique of interpolation systems [*Numer. Math.* 39:1–14 (1982)], when it is applied to prove the unisolvence of those interpolation problems. The solution of the problem is given in a Newton form.

1. INTRODUCTION

Recently, Mühlbach and Reimers [3] have computed Cauchy-Vandermonde determinants, including as particular cases Vandermonde (confluent or not) and Cauchy determinants. Those determinants arise in some rational interpolation problems in one variable.

On the other hand, in [2] we have computed multidimensional Vandermonde determinants using the technique of interpolation systems (cf. [1]).

In this paper we use the same technique to extend some of the results of Mühlbach and Reimers, proving the existence and uniqueness of solution of certain bivariate rational interpolation problems. These solutions are given in a Newton-like form, since they are obtained from a triangular linear system. The determinant being triangular, its value can be directly computed too.

*Supported by D. G. Aragón Research Grant CB3/85.

655 Avenue of the Americas, New York, NY 10010 0024-3795/89/$3.50

2. ON THE SOLUTION OF SOME BIVARIATE RATIONAL INTERPOLATION PROBLEMS

The Lagrange interpolation problem in $\mathbb{R}^2$ is defined by a finite set S of points (interpolation points) and a vector space V of bivariate functions (interpolation space). We look for a function $p \in V$ taking given values on S.

S can always be described in the form

$$S = \left\{ u_{ij} = r_i \cap r_{ij} \middle| (i, j) \in I \right\}, \tag{1}$$

where r_i, r_{ij} are, respectively, the straight lines of the equations $x - x_i = 0$, $y - y_{ij} = 0$, and I is the index set

$$I = \{(i, j) \mid i = 0, 1, \ldots, n;\ j = 0, 1, \ldots, m(i)\} \tag{2}$$

lexicographically ordered. Without restriction, we assume $x_i \neq x_h$ if $i \neq h$, $y_{ij} \neq y_{ik}$ if $j \neq k$, and

$$m(0) \geqslant m(1) \geqslant \cdots \geqslant m(n). \tag{3}$$

Consider the set of univariate functions

$$A = \left\{ q_1, q_1^2, \ldots, q_1^{n_1}, q_2, q_2^2, \ldots, q_M, q_M^2, \ldots, q_M^{n_M}, 1, x, \ldots, x^{n_0} \right\}, \tag{4}$$

where $q_i(x) = (x - b_i)^{-1}$, $b_1, b_2, \ldots, b_M$ are pairwise distinct real numbers such that $\{x_0, x_1, \ldots, x_n\} \cap \{b_1, b_2, \ldots, b_M\} = \varnothing$, and $M \geqslant 0$, $n_0 \geqslant -1$, $n_i \geqslant 1$ $(i = 1, 2, \ldots, M)$ are integers with

$$n_0 + n_1 + \cdots + n_M = n. \tag{5}$$

Note that $M = 0$ (respectively $n_0 = -1$) means that there is no function of type q_i^j (respectively x^j) in A. The set A spans the space of rational functions with denominator

$$\prod_{j=1}^{M} (x - b_j)^{n_j}$$

and numerator of degree not greater than n. As usual, empty products are taken as 1 and empty sums as 0.

Analogously, we consider the set

$$B = \left\{ \tilde{q}_1, \tilde{q}_1^2, \ldots, \tilde{q}_1^{m_1}, \tilde{q}_2, \tilde{q}_2^2, \ldots, \tilde{q}_N, \tilde{q}_N^2, \ldots, \tilde{q}_N^{m_N}, 1, y, \ldots, y^{m_0} \right\}, \tag{6}$$

where $\tilde{q}_j(y) = (y - c_j)^{-1}$, $c_1, c_2, \ldots, c_N$ are pairwise distinct real numbers such that $\{y_{ij} | (i,j) \in I\} \cap \{c_1, c_2, \ldots, c_N\} = \varnothing$, and $N \geqslant 0$, $m_0 \geqslant -1$, $m_j \geqslant 1$ $(j = 1, 2, \ldots, N)$ are integers with

$$m_0 + m_1 + \cdots + m_N = m(0). \tag{7}$$

For simplicity, we denote by $\varphi_0, \varphi_1, \ldots, \varphi_n$ the functions of A and $\psi_0, \psi_1, \ldots, \psi_{m(0)}$ those of B, with the same ordering as in (4) and (6), respectively.

Our interpolation space will be the space V spanned by

$$C = \{\phi_{ij} = \varphi_i \psi_j | (i,j) \in I\}. \tag{8}$$

There exists a wide variety of possibilities for the space V according to the choice of M, N, n_i, m_i: only polynomials ($M = N = 0$), only simple poles ($n_i = 1$ $\forall i \geqslant 1$, $m_j = 1$ $\forall j \geqslant 1$), only nonpolynomial rational functions ($n_0 = m_0 = -1$), etc.

The interpolation problem can be stated as:

(P) Find $\phi \in V$ such that

$$\phi(u_{ij}) = z_{ij} \qquad \forall (i,j) \in I,$$

where z_{ij} are given real numbers.

Since the functions of A (and similarly those of B) are linearly independent as a consequence, for example, of the regularity of the matrix $(\varphi_i(x_j))_{i,j=0,\ldots,n}$ (cf. [3]), then the functions of C are linearly independent and therefore C is a basis of V.

The unisolvence of the problem (P) is easily proved by using a new basis $D = \{F_{ij} | (i,j) \in I\}$, which is a natural extension of the basis

$$\left\{ \prod_{k=0}^{i-1} (x - x_k) \right\}_{i=0}^{n}$$

of P_n used in the Newton formula for one variable, and similar to the basis used in [1]. We denote by F_{ij} the bivariate function

$$F_{ij}(x,y) = \frac{\prod_{h=0}^{i-1} (x - x_h) \cdot \prod_{k=0}^{j-1} (y - y_{ik})}{Q_i(x) \tilde{Q}_j(y)}, \tag{9}$$

where $Q_i(x)$ is defined in the following way:

$$Q_i(x) = \prod_{j=1}^{M} (x - b_j)^{n_j} \qquad \text{if} \quad i \geqslant n_1 + \cdots + n_M, \tag{10}$$

$$Q_i(x) = (x - b_{k+1})^{h+1} \prod_{j=1}^{k} (x - b_j)^{n_j}$$

$$\text{if} \quad i = n_1 + \cdots + n_k + h, \ 0 \leqslant h < n_{k+1}, \quad k < M, \tag{10'}$$

and analogously for $\tilde{Q}_j(y)$ with N, m_j, y, c_j instead of M, n_j, x, b_j.

Observe that

$$\deg Q_i = \begin{cases} n_1 + n_2 + \cdots + n_M & \text{if} \quad i \geqslant n_1 + n_2 + \cdots + n_M, \\ i + 1 & \text{if} \quad i < n_1 + n_2 + \cdots + n_M, \end{cases} \tag{11}$$

and therefore the difference between the x-degree of the denominator and that of the numerator of F_{ij} is

$$\begin{cases} -k & \text{if} \quad i = n_1 + \cdots + n_M + k, \quad k \geqslant 0, \\ 1 & \text{if} \quad i < n_1 + \cdots + n_M. \end{cases} \tag{12}$$

Analogously for the y-degree. The numerator of F_{ij} has been constructed so that we have

$$F_{ij}(u_{hk}) = 0 \qquad \text{if} \quad (i, j) > (h, k) \quad [(i, j), (h, k) \in I],$$

$$F_{ij}(u_{ij}) \neq 0 \qquad \forall (i, j) \in I. \tag{13}$$

THEOREM.

(i) $D = \{ F_{ij} \mid (i, j) \in I \}$ *is a basis of* V.

(ii) *The matrix of change from D to C (or vice versa) is triangular with its diagonal entries different from zero.*

(iii) *The problem* (P) *has a unique solution.*

(iv) *The solution of the problem can be written*

$$p = \sum_{(i, j) \in I} a_{ij} F_{ij}, \tag{14}$$

where

$$a_{ij} = \frac{z_{ij} - \sum_{(h,k)<(i,j),\,(h,k)\in I} a_{hk} F_{hk}(u_{ij})}{F_{ij}(u_{ij})}. \tag{15}$$

Proof. (i): Suppose

$$\sum_{(i,j)\in I} \lambda_{ij} F_{ij} = 0. \tag{16}$$

Evaluating (16) for each $u_{hk} = (x_h, y_{hk})$, $(h,k)\in I$, we get

$$F\Lambda = 0, \tag{17}$$

where $F = (F_{ij}(u_{hk}))_{(i,j),(h,k)\in I} \in \mathbb{R}^{\nu\times\nu}$, $\Lambda = (\lambda_{ij})^T_{(i,j)\in I} \in \mathbb{R}^{\nu}$, ν being the cardinality of the set I. Since due to (13) F is triangular with nonzero diagonal entries, $\Lambda = 0$. Therefore D is a basis of V.

(ii): Let us write (9) in the form

$$F_{ij}(x,y) = f_i(x) f_{ij}(y). \tag{18}$$

Each factor can be decomposed in partial fractions (after division if the numerator degree is greater or equal than the denominator degree). Multiplying the two decompositions, we express F_{ij} as a linear combination of functions $\phi_{hk}\in C$ with $(h,k)\leqslant(i,j)$ (in fact, with $h\leqslant i$ and $k\leqslant j$), and therefore the matrix of change is triangular. We write

$$F_{ij}(x,y) = \sum_{\substack{(h,k)\in I\\(h,k)\leqslant(i,j)}} \alpha_{hk}^{(i,j)} \phi_{hk}. \tag{19}$$

Note that, as mentioned above, a polynomial could appear in the decomposition due to division, but this occurs only if $i\geqslant n_1+n_2+\cdots+n_M$ [or $j\geqslant m_1+m_2+\cdots+m_N$], and then $\varphi_i(x)$ [or $\Psi_j(y)$] in ϕ_{ij} is a polynomial too. Since D and C are bases of V, the regularity of the matrix of change implies that its diagonal entries are different from zero.

(iii) and (iv) are a direct consequence of (13).

REMARK. (14) can be called a Newton formula for (P), since (15) is similar to the well-known formula for classical divided differences.

3. AN EXAMPLE OF APPLICATION: COMPUTATION OF SOME BIVARIATE CAUCHY-VANDERMONDE DETERMINANTS

Determinants of the type $\det(\phi_{ij}(u_{hk}))_{(i,j),(h,k)\in I}$ can be called bivariate Cauchy-Vandermonde determinants (cf. [3] for the univariate case). According to the above theorem, their value is easily obtained by means of the formula

$$\det\big(\phi_{ij}(u_{hk})\big)_{(i,j),(h,k)\in I} = \frac{\prod_{(i,j)\in I} F_{ij}(u_{ij})}{\prod_{(i,j)\in I} \alpha_{ij}^{(i,j)}}, \tag{20}$$

where $\alpha_{ij}^{(i,j)}$ is the coefficient of ϕ_{ij} in (19).

The wide variety of possibilities of the space V, due to different choices of M, N, m_i, n_i, and the different computation of $\alpha_{ij}^{(i,j)}$ for each case would make the notation too cumbersome for the computation of (20) in the general case. As an example, we compute it for the particular case $n_i = m_i = 1\ \forall i \geqslant 1$, $n_0 = m_0 = -1$. In this case, there is no polynomial in V. That is, the determinant we will compute is

$$\det\left(\frac{1}{(x_i - b_h)(y_{ij} - c_k)}\right), \tag{21}$$

with $(i,j)\in I$ (row index) and $(h,k)\in J$ (column index), with $J = \{(h'+1, k'+1) \mid (h',k')\in I\}$.

Reducing to the common denominator in the equation (19) and evaluating at (b_{h+1}, c_{k+1}), we easily get

$$\alpha_{hk}^{(i,j)} = \begin{cases} \dfrac{\prod_{p=0}^{i-1}(b_{h+1} - x_p)\prod_{q=0}^{j-1}(c_{k+1} - y_{iq})}{\prod_{\substack{l=1\\ l\neq h+1}}^{i+1}(b_{h+1} - b_l)\prod_{\substack{m=1\\ m\neq k+1}}^{j+1}(c_{k+1} - c_m)} & \text{if} \quad h \leqslant i, k \leqslant j, \\ 0 & \text{if} \quad h \leqslant i, k > j, \end{cases} \tag{22}$$

and therefore, taking into account (9) and (10), we have

$$\det\big(\phi_{ij}(u_{hk})\big)_{(i,j),(h,k)\in I} = U/L,$$

where

$$U=\prod_{i=0}^{n}\left[\prod_{j=0}^{m(i)}\left(\frac{\prod_{p=0}^{i-1}(x_i-x_p)\prod_{q=0}^{j-1}(y_{ij}-y_{iq})}{\prod_{l=1}^{i+1}(x_i-b_l)\prod_{m=1}^{j+1}(y_{ij}-c_m)}\right)\right]$$

and

$$L=\prod_{i=0}^{n}\left[\prod_{j=0}^{m(i)}\left(\frac{\sum_{p=0}^{i-1}(b_{i+1}-x_p)\prod_{q=0}^{j-1}(c_{j+1}-y_{iq})}{\prod_{l=1}^{i}(b_{i+1}-b_l)\prod_{m=1}^{j}(c_{j+1}-c_m)}\right)\right].$$

Other cases can be treated similarly.

REFERENCES

1 M. Gasca and J. I. Maeztu, On Lagrange and Hermite interpolation in $\mathbb{R}^k$, *Numer. Math.* 39:1–14 (1982).

2 M. Gasca and J. J. Martínez, On the computation of multivariate confluent Vandermonde determinants and its applications, in *The Mathematics of Surfaces II* (R. Martin, Ed.), Oxford U.P., 1987.

3 G. Mülbach and L. Reimers, Linear extrapolation by rational functions, exponentials and logarithmic functions, *J. Comput. Appl. Math.* 17:329–344 (1987).

4 H. Werner, Remarks on Newton type multivariate interpolation for subsets of grids, *Computing* 25:181–191 (1980).

Received 15 December 1987; final manuscript accepted 28 May 1988